Local Galois Module Structure in Prime Characteristic and Galois Scaffolds

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Notation

Let K be a local field with perfect residue field both of prime characteristic, p.

Consider L/K a finite Galois extension of degree p^n, $n \geq 1$ and Gal(L/K)=G.

Also let $\mathfrak{O}_L$ and $\mathfrak{O}_K$ be their respective valuation rings with unique maximal ideals $\mathfrak{P}_L$ and $\mathfrak{P}_K$.

Denote $\mathfrak{A}_{L/K}$ to be the associated order of $\mathfrak{O}_L$.

Recall the ramification groups of L/K defined as

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{P}_L^{i+1}, \forall x \in \mathfrak{O}_L\} \text{ for } i \in \mathbb{Z}_{\geq -1}.$$

Here we consider totally ramified extensions with ramification numbers $b_1 \leq \cdots \leq b_n$ with $(b_i, p) = 1$.

Cyclic Extensions of Degree p.

Let L/K be a cyclic extension of degree p with a unique ramification number b and let $a = rem(p, b)$.

- $\mathfrak{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if $a|(p-1)$ (Aiba, 2003).

Let us also consider a more general notion of freeness of fractional ideals of L - $\mathfrak{P}_L^h$ over their respective associated order defined as

$$\mathfrak{A}_{L/K}(h) = \{\alpha \in K[G] : \alpha\mathfrak{P}_L^h \subseteq \mathfrak{P}_L^h\} \text{ for } h \in \mathbb{Z}.$$

- Theorem by Ferton (1973) states exactly which $\mathfrak{P}_L^h$ are free in char$(K) = 0$. It has been shown independently by Marklove (2014) and Huynh to work in char$(K) = p$.
- De Smit and Thomas (2007) used continued fractions to count minimal number of $\mathfrak{A}_{L/K}$-generators.

Galois Scafford

Set $\mathbb{S}_p = \{0, 1, \ldots, p-1\}$ and $\mathbb{S}_{p^n} = \{0, 1, \ldots, p^n - 1\}$. We write $s \in \mathbb{S}_{p^n}$ as $s = \sum_{i=0}^{n-1} s_{(i)} p^i$.

Let L/K be as before of degree p^n. Assume $b_i \equiv b_j \pmod{p^n}$ for all i, j. Denote $b \equiv b_n \pmod{p^n}$.

For $t \in \mathbb{S}_{p^n}$ define a map $\mathfrak{a} : \mathbb{S}_{p^n} \to \mathbb{S}_{p^n}$ by $\mathfrak{a}(t) := -b^{-1} t \pmod{p^n}$.

Definition: A *Galois scaffold* on L/K (of tolerance $\mathfrak{T} = \infty$) comprises of :

(1) elements $\lambda_t \in L$ for $t \in \mathbb{Z}$ such that $v_L(\lambda_t) = t$.

(2) $\Psi_i \in K[G]$ for $1 \leq i \leq n$ such that $\Psi_i 1 = 0$ and such that for each i and for each $t \in \mathbb{Z}$ we have

$$\Psi_i \cdot \lambda_t = \begin{cases} \lambda_{t + p^{n-i} b} & \text{if } \mathfrak{a}(t)_{(n-i)} \geq 1, \\ 0 & \text{if } \mathfrak{a}(t)_{(n-i)} = 0. \end{cases}$$

Are there any ramified extensions that possess a Galois scaffold?
Yes. For exapmle, *one-dimensional* and *near one-dimensional elementary abelian extensions*.

Galois Scaffolds 2

For $s \in \mathbb{S}_{p^n}$ write $s \preceq u$ if $s_{(i)} \leq u_{(i)}$ for all $0 \leq i \leq n-1$. Let $1 \leq b \leq p^n - 1$ and $b - p^n + 1 \leq h \leq b$.
Define

$$d(s) = \left\lfloor \frac{b(s+1) - h}{p^n} \right\rfloor$$

and

$$w(s) = \min\{d(u) - d(u-s) : u \in \mathbb{S}_{p^n}, s \preceq u\},$$

which is equivalent to

$$w(s) = \min\{d(s+j) - d(j) : j \in \mathbb{S}_{p^n}, j \preceq p^n - 1 - s\}.$$

Theorem (Byott & Elder,2014, partial)

Suppose L/K has a Galois scaffold. Then $\mathfrak{P}_L^h$ is free over $\mathfrak{A}_{L/K}(h)$ if and only if $w(s) = d(s)$ for all $s \in \mathbb{S}_{p^n}$.

Ferton in characteristic p

Theorem

Let L/K be totally ramified cyclic extension of degree p with ramification number b and $b-p+1 \leq h \leq b$. Also let b/p have a continued fraction expansion $[0; q_1, q_2, \ldots, q_n]$ of length n, with $q_n \geq 2$.

(i) *If $b = 1$, then $\mathfrak{P}_L^h$ is free over $\mathfrak{A}_{L/K}(h)$ iff $\frac{1-p}{2} \leq h \leq 0$. Taking $p > 2$ this inequality becomes $\frac{3-p}{2} \leq h \leq 1$.*

(ii) *If $b > 1$ and $b \geq h \geq 0$, then $\mathfrak{P}_L^h$ is free over $\mathfrak{A}_{L/K}(h)$ iff*

  (a) *for even n, $h = b$ or $h = b - q_n$,*

  (b) *for odd n, $b - (q_n/2) \leq h \leq b$.*

(iii) *If $b > 1$ and $h < 0$, then $\mathfrak{P}_L^h$ is not free over $\mathfrak{A}_{L/K}(h)$.*

Corollaries to the Byott & Elder Theorem

- Ferton: By replacing p with p^n in the fraction expansion we get: free ideals in degree p^n = Ferton + additional ones.
- $\mathfrak{P}_L^b$ is free over $\mathfrak{A}_{L/K}(h)$ for all b.
- If $b = p^n - 1$, then $\mathfrak{P}_L^h$ is free $\iff h = p^n - p^k$ for $0 \leq k \leq n$.
- If $b = p^n - 2$ with $p \neq 2$, then precisely $\mathfrak{P}_L^b$ and $\mathfrak{P}_L^{b-1}$ are free.
- If $b = (p-1)/m$ with $1 \leq m \leq p-1$ and $m \mid (p-1)$, then for $0 \leq h \leq (p-1)/m$, $\mathfrak{P}_L^h$ is free $\iff h = 0$ or $h = b$.
- If $n = 2$, $b = p+1$ with $p \neq 2$, then $\mathfrak{P}_L^h$ is free $\iff h = 0$ or $h \geq (p+1)/2$.